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Optimal running trajectory in Baseball

INMA 2471 – Optimization models and methods II
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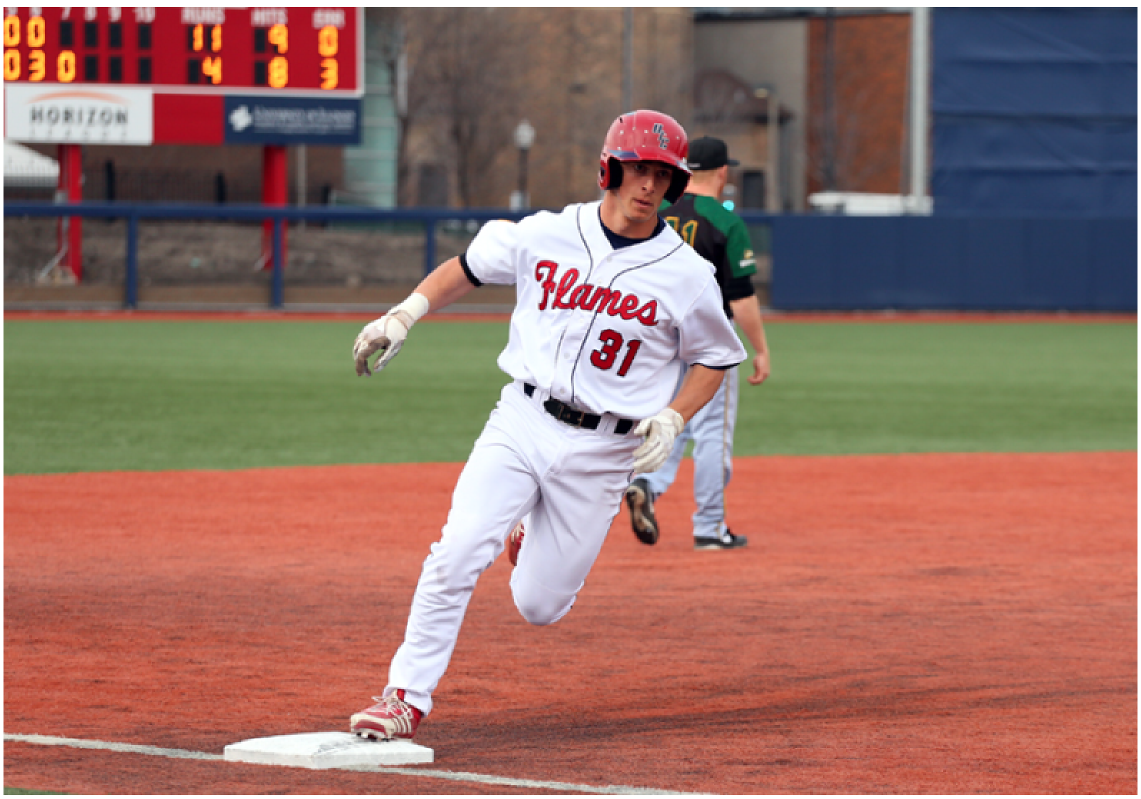

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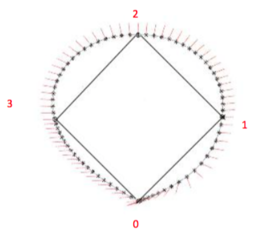
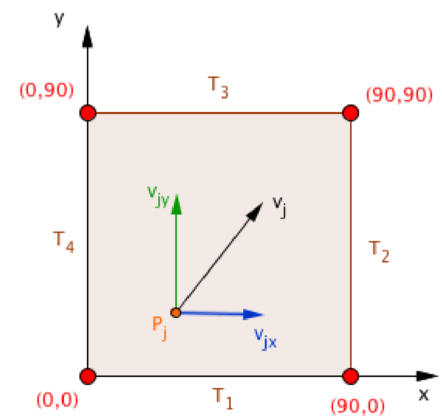
1 Introduction and description of the model



In this problem, we are asked to estimate the optimal path which minimizes the time taken for a player to go through all four bases and analyze the nature of our solution. Some basic constraints will have to be always satisfied : **1)** the runner start on base 1 and has a zero initial velocity. **2)** During the path, the maximum acceleration is limited to 10 feet/sec. **3)** The distance between two bases is 90 feet. We will also consider the bases' order as showed on the figure. As it is almost impossible to deal with the real continuous problem, we will approximate it by solving a discrete time model: $(t_1.....t_n)$: this means that we will consider all the physical parameters as a constant number between two instant of time : t_i and t_{i+1}. For example instead of a continuous real function $a(t)$, we will describe the acceleration as a series of real numbers $a_1, \ldots, a_{n-1}$ which all have to satisfy the general constraint on the acceleration. Thus the variables (acceleration, speed, distance) are presented by (a_i, v_i, s_i) with subscript $i = 1 \ldots n$.

2 Modeling



We are working in a two dimension problem, it is then natural to decompose all physical variables that are vectors, in their x and y composants. This enables us to work with real numbers instead of vectors : in our model we will consider that one side of the baseball ground concide with the axe x, this doesn't change anything to the solution.

We will divide our problem in 4 subproblems : one subproblem per segment. Thus four time variables (T_1, T_2, T_3, T_4) are added to represent the time taken between two adjacent bases (T_4 represents the time to come back at base 0 from base 3). Following the idea evoked in the introduction, we will consider $n-1$ discretization points for each subproblems. Thus each adjacent points of subproblem i are separated by a time $t_i^{step} = \frac{T_i}{(n-1)}$. To summarize, we get a series $(t_1.....t_{4n-3})$ as time variables, and we want to minimize :

$$f_{obj} := t_{4n-3} = T = \sum_i T_i$$

For each discretization point P_j correspond values for acceleration $a_j = a(t_j)$, velocity $v_j = v(t_j)$ and displacement $s_j = s(t_j)$. As we have mentioned in the introduction some basic constraints must be satisfied[1] :

Initial velocity	$v_x(t_1) = v_y(t_1) = 0$	
Limited acceleration	$\forall j : \quad a_x^2(t_j) + a_y^2(t_j) \leq 10^2$	
Bases	$(s_x(t_n), s_y(t_n)) = (90, 0)$	$(s_x(t_{2n-1}), s_y(t_{2n-1})) = (90, 90)$
	$(s_x(t_{3n-2}), s_y(t_{3n-2})) = (0, 90)$	$(s_x(t_{4n-3}), s_y(t_{4n-3})) = (0, 0)$

In order to reduce the number of variables of our model we will only keep **velocity variables**. It is easy to express acceleration and distance in terms of velocity based on the fact that $a(t) = \Delta v / \Delta t$ and that $s(t) = s_0 + v_0 * t + \frac{a_0 * t^2}{2}$. For example, if we consider the discretization $P(t_j)$ where $t_j \in [t_n, t_{2n-1}]$, the variables are :

$$\begin{aligned}
a_{x,j} &= \frac{v_{x,j+1} - v_{x,j}}{t_2^{step}} \\
s_{x,j} &= s_{x,j-1} + v_{x,j-1} * t_2^{step} + \frac{a_{x,j-1}}{2} * (t_2^{step})^2 = s_{x,j-1} + v_{x,j-1} * t_2^{step} + \frac{v_{x,j} - v_{x,j-1}}{2 * t_2^{step}} * (t_2^{step})^2 \\
&= s_{x,j-1} + (v_{x,j-1} + v_{x,j}) * \frac{t_2^{step}}{2} \\
&= s_{x,t_n} + (v_{x,t_n} + v_{x,j}) * \frac{t_2^{step}}{2} + t_2^{step} * \sum_{k=t_{n+1}}^{j-1} v_{x,k}
\end{aligned}$$

[1] We have to mention that with our discretization we have that : $T_1 = t_n$, $T_1 + T_2 = t_{2n-1}$, $T_1 + T_2 + T_3 = t_{3n-2}$ and $T_1 + T_2 + T_3 + T_4 = t_{4n-3}$

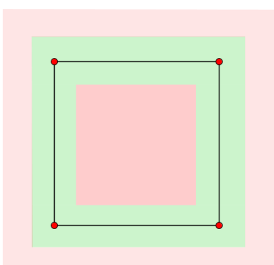
The previous constraints become then :

$$\begin{array}{rll}
\textbf{Initial velocity} & v_x(t_1) = v_y(t_1) = 0 & \\
\textbf{Limited acceleration} & \forall t_j \in subproblem_i : \quad (v_{x,\,j+1} - v_{x,\,j})^2 + (v_{y,\,j+1} - v_{y,\,j})^2 \leq 10^2 \cdot (t_i^{step})^2 & \\
\textbf{Bases} & (s_x(t_n), s_y(t_n)) = (90, 0) & (s_x(t_{2n-1}), s_y(t_{2n-1})) = (90, 90) \\
 & (s_x(t_{3n-2}), s_y(t_{3n-2})) = (0, 90) & (s_x(t_{4n-3}), s_y(t_{4n-3})) = (0, 0)
\end{array}$$

where :

$$\begin{aligned}
s_x(t_n) &= s_{x,\,t_1} + \frac{t_1^{step}}{2} * (v_{x,\,t_1} + v_{x,\,t_n}) + t_1^{step} * \sum_{k=2}^{t_{n-1}} v_{x,\,k} = t_1^{step} \cdot \left(0.5 \cdot v_{x,\,t_n} + \sum_{k=2}^{t_{n-1}} v_{x,\,k}\right) \\
s_x(t_{2n-1}) &= s_{x,\,t_n} + \frac{t_2^{step}}{2} \cdot (v_{x,\,t_n} + v_{x,\,t_{2n-1}}) + t_2^{step} * \sum_{k=t_{n+1}}^{t_{2n-2}} v_{x,\,k} \\
&\vdots \\
s_y(t_{4n-3}) &= s_{x,\,t_{3n-2}} + \frac{t_4^{step}}{2} \cdot (v_{x,\,t_{3n-2}} + v_{x,\,t_{4n-3}}) + t_4^{step} * \sum_{k=t_{3n-1}}^{t_{4n-4}} v_{x,\,k}
\end{aligned}$$

To add some realism to the model, we will consider some additional constraints :



1) a maximum velocity for the runner
2) a inferior maximum value for deceleration than for acceleration ($d_{max} \leq a_{max}$)
3) the 3 feet rule : a players can not go further than 3 feet from the lines of the square formed by the bases.

Again all theses constraints can be written as an expression of the $8n - 2$ variables that follows : $(T_1, T_2, T_3, T_4, v_{x,1} \dots v_{x,4n-3}, v_{y,1} \dots v_{y,4n-3})$:

$$\begin{array}{rl}
\textbf{Limited velocity} & \forall t_j \quad v_{y,\,j}^2 + v_{y,\,j}^2 \leq v_{max}^2 \\
\textbf{Limited acceleration} & if\ ||\mathbf{v_{j+1}}|| > ||\mathbf{v_j}|| : \ \forall t_j \in subproblem_i : \quad (v_{x,\,j+1} - v_{x,\,j})^2 + (v_{y,\,j+1} - v_{y,\,j})^2 \leq \mathbf{a_{max}^2} \cdot (t_i^{step})^2 \\
\textbf{Limited deceleration} & if\ ||\mathbf{v_{j+1}}|| > ||\mathbf{v_j}|| : \ \forall t_j \in subproblem_i : \quad (v_{x,\,j+1} - v_{x,\,j})^2 + (v_{y,\,j+1} - v_{y,\,j})^2 \leq \mathbf{d_{max}^2} \cdot (t_i^{step})^2 \\
\textbf{3 feet} & \text{Subprob 1:} \quad -3 \leq s_{x,i} \leq 93 \text{ and } -3 \leq s_{y,i} \leq 3 \quad \text{Subprob 2:} \quad 87 \leq s_{x,i} \leq 93 \text{ and } -3 \leq s_{y,i} \leq 93 \\
 & \text{Subprob 3:} \quad -3 \leq s_{x,i} \leq 93 \text{ and } 87 \leq s_{y,i} \leq 93 \quad \text{Subprob 4:} \quad -3 \leq s_{x,i} \leq 3 \text{ and } -3 \leq s_{y,i} \leq 93
\end{array}$$

3 Analysis of the model

3.1 Study of convexity

Even though the description of the constraints is simple, those are not necessarily linear or convex. For example the limited acceleration constraint $\frac{(v_{x,i+1} - v_{x,i})^2}{\triangle t_1^2} + \frac{(v_{y,i+1} - v_{y,i})^2}{\triangle t_1^2} \leq 100$ is not convex . Indeed let f(x,y,t)=$\frac{(x-y)^2}{t^2}$, we will study the convexity of function f by giving the Hermitian matrix.

$$\triangle f = \begin{pmatrix} \frac{2}{t^2} & -\frac{2}{t^2} & \frac{4x}{t^3} \\ -\frac{2}{t^2} & \frac{2}{t^2} & \frac{4y}{t^3} \\ \frac{4x}{t^3} & \frac{4y}{t^3} & \frac{6(x-y)^2}{t^4} \end{pmatrix}$$

Unfortunately, it is not a defined positive matrix (ex: $x = y = 10$, $t = 0.5$, we have an eigenvalue -8.4853). In the same way, we can prove that the bases constraints are also not convex, thus our model is not convex. We can then not affirm that the solution given with AMPL is a global optimal solution, even for the discretize model. All solutions are local optimum and can be a global one but it is not guarentee. Thus all solutions given by AMPL are **upper bound** of the optimal solution if there exits one.

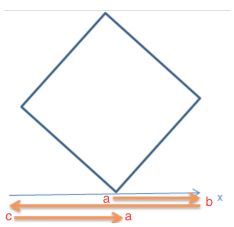
3.2 Convergence to the optimal solution

When the number of discretization points n tends to infinite, the optimal solution of our simplified model converge to the real optimal value. We note T_p the optimal real time taken to run along a certain path p. For n fixed, $T_{p(n)}$ represents the time taken by a discrete path with n discretization points for each subproblems. We suppose that there is convergence(which can be mathematically proved), we have that :

$$T_p = \lim_{n \to \infty} T_{p(n)} = \min_n T_{p(n)}$$

Actually, every $T_{p(n)}$ is an **upper bound** of the optimal value because all discretize path are physically realizable : our player will have a constant acceleration between t_j and t_{j+1}.

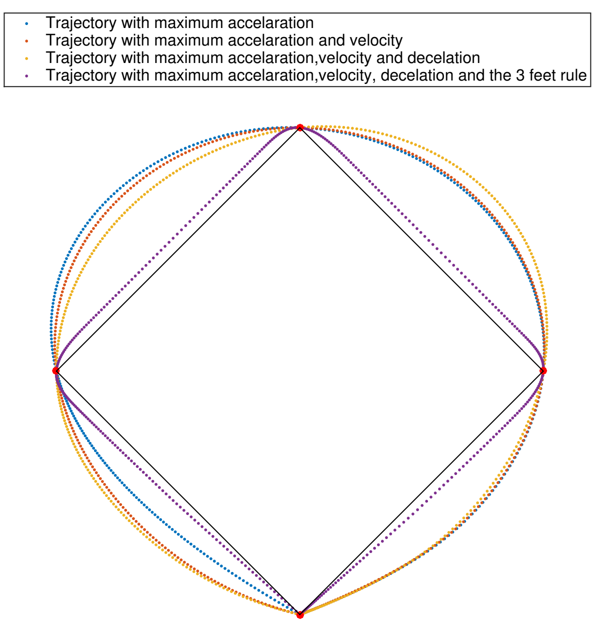
3.3 Lower Bound



We will give two different lower bounds in this section :

1) If the player has to run $90 \cdot 4$=360 feet on a straight line with an acceleration limited to 10 feet/s^2, he will take at least 8.4853 seconds. Using the fact that $s_x = \frac{a_{max}t^2}{2}$.
2) The path projected on the x axis can be represented in three periods : $a \to b$, $b \to c$, $c \to a$ where $a = (0,0)$, $b = (s,0)$ and $c = (-s,0)$ with s=$\frac{90}{\sqrt{2}}$.
We assume that at the end of the first period the player should arrive at point b with a zero-speed : if he arrives at b with a positive speed, he will go farther than b and that can't be a optimal solution. The optimal path is to accelerate at a_{max} from point a until the middle point $(s/2, 0)$ and then decelerate to point b. Let t_1 be the time for the player to reach the middle point. It takes exactly the same time for the player to reach point b from the middle point. Thus we have $0.5 \cdot a_{max}t_1^2 + 0.5 \cdot a_{max}t_1^2 = s$, where $a_{max} = 10$. We get $t_1 = 2.52$ seconds so $t_{a \to b} = 2t_1$. Using the same method for $b \to c$ we get $t_2 = 3.57$ seconds and $t_{b \to c} = 2t_2$. Finally for $c \to a$, it is logical to always accelerate without decelerate : we get then $t_{c \to a} = 3.57$ seconds. Putting all together gives us a **lower bound** equal to 15.54 seconds.

4 Analysis of the results



In our model will consider the following set of parameters : $a_{max} = 10$ fs^{-2}, $d_{max} = 8$ fs^{-2} and $v_{max} = 30$ fs^{-1}.

First of all, we can remark that the trajectory with only a maximum acceleration is very close from the path given in the reference. As the intuition would have told us, to go the faster as possible, the runner has to adopt a curve trajectory. With $n = 100$ discretization points per subproblem, we get a optimal value of $T = 16.5793$ seconds. We beat then by 0.13 seconds the affirmation of the reference.

We are all humans, but we are not all Usain Bolt who has a maximum speed of 40.7 fs^{-1} on a 100 meter. Thus we think that it is wise to consider a maximum velocity of 30 fs^{-1}. On the figure here next, the trajectory is the orange one. We can observe that it is very close from the previous one (blue), however when there was no limit to the velocity, the strategy was the take distance from segment 3 to have the longest "straight" path at the end and the accelerate with maximal acceleration. With a maximal velocity value, it is no more possible to act in this manner, therefore the runner run closer from segment 3. With this additional constraint we get a optimal value of $T = 16.7439$ seconds. We conclude that the constraint was not too restrictive and didn't impact much the optimal solution.

In reality we can not decelerate as fast as we accelerate, so in order to improve again the realism of our model, we have decided to consider a maximum deceleration value. We have considered that there is deceleration if the norm of the velocity decreases between t_j and t_{j+1}. On the figure here above, the trajectory is the yellow one. The trajectory seems logical : indeed the runner will have to decelerate mainly in the turns. We can see that the trajectory is nearly the same than the previous case on segment 1, but the runner have to take a longer turn on the first base. This respects our intuition. With this additional constraint we get a optimal value of $T = 17.4814$ seconds.

Finally we can directly see that the 3 feet rule is the most restrictive : the trajectory is the purple one. In the previous cases, the strategy was to adopt a curved trajectory. Here because of the 3 feet constraint such a trajectory is not allowed, however we can see that the runner will still adopt a curve trajectory on the corner which seems logical. In this last case the optimal time to run a home run is $T = 20.9966$ seconds.

What we did from the beginning of this section is to add each time one more constraint to get closer to the reality. Our model is then more and more constrained : the optimal solution of case i will always be a upper bound for case $i+1$. This is what we observe : the more constraints we have, the longer will it take to run a home run :

Case	**Optimal Time**
maxAcceleration	16.5793
maxAcceleration + maxVelocity	16.7439
maxAcceleration + maxVelocity + maxDeceleration	17.4814
maxAcceleration + maxVelocity + maxDeceleration + 3feet	20.9966

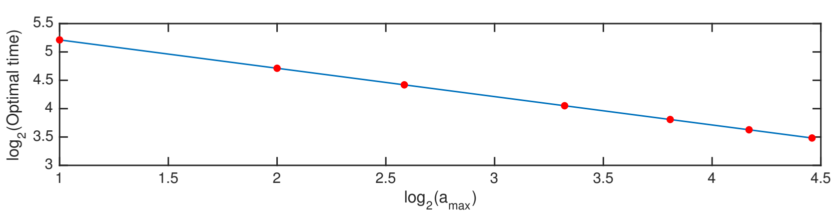
4.1 Influence of the number of discretization points

In this subsection we will consider only the basic constraints to study the impact of the number of discretization points on the optimal time obtained. We hope that these results will converged as we said in subsection **3.2**. We see that the optimal time tend to stay around 16.579 seconds. Based on we we have said in subsection **3.2** and **3.3**, we can affirm that the optimal solution is in the interval [15.54, 16.5791] seconds.

n	**Optimal Time** (s)
5	16.6580
10	16.5937
100	16.5793
1000	16.5792
10000	16.5791

4.2 Sensibility of our model to the maximum acceleration value

Again we consider here only the basic constraints. The aim of this subsection is to have a look the the optimal time when we vary the maximum acceleration value from 2 to 22 fs^{-2} with a number of $n = 50$ discretization points. If we plot the path for each of these values, we can remark that it stays the same : this is explain by the fact that there is no maximum value on the velocity. Indeed if we increase the maximum value of the acceleration limit, the runner will run the same path by quicker. If we plot the $log_2 - log_2$ graph of the values obtained in the table. We see that all data are on a line of steep -0.5. This explains the fact that the model is more sensible with variations of a_{max} where a_{max} is small.



$\mathbf{a_{max}}$ (f/s^2)	**Optimal Time** (s)
2	37.0732
4	26.2147
6	21.4042
10	16.5796
14	14.0124
18	12.3577
22	11.1780

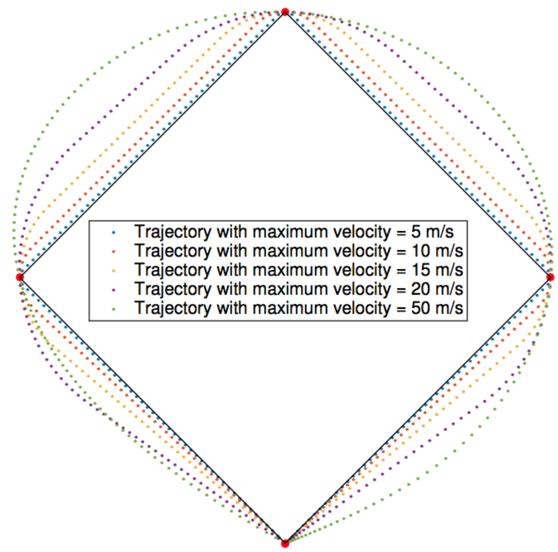
4.3 Sensibility of our model to the maximum velocity value

In this subsection we will consider the basic constraints and the maximum velocity constraint. The parameters used are $a_{max} = 10$ and $n = 50$. The results are the following :

$\mathbf{v_{max}}$ (f/s)	**Opt. Time** (s)
5	72.4744
10	36.9762
15	25.4904
20	20.0369
50	16.5796

We can analyse the graphical result and compare it to our intuition. Evidently the quicker a player can run the less time he will take to accomplish a home run. Moreover, the player acceleration is limited to 10 fs^{-2} then if he runs at 5 f/s he is able to stop completely in the direction he was running and to run at 5 f/s in an orthogonal direction. This is what exactly we see on the graph : the players that is limited to 5 f/s will follow the line of the square : indeed he can take the turns at maximum velocity. Based on the same idea, the higher will be the maximum velocity, the less sharp the players will be able to take the corners and the rounder will be his trajectory[a]. This is what we see on the graph here next.

[a] If we consider that he will run at maximum velocity. He has always the possibility to run at lower velocity but this won't lead to the optimal solution.



5 Conclusion

Our model is unfortunately not convex, thus all solutions are not guarantee to be globally optimal. However we simple calculations we have been able to define an interval where the optimal solution belongs. With all the constraints the model tends to be as close from the reality as possible. However it is always possible to make it even closer. For example we could consider that the player has a push of adrenaline when he begins to run the last segment ! We could then consider a higher maximum velocity like 35 fs^{-1}. If we compute this situation with all other parameters identical as in section **4** we obtain an optimal time of 16.6178 seconds which is inferior to the case where we consider only one maximum velocity value : 16.7439 seconds.

There is no end to this model and we could always endorse it. However the changes will less and less affect the optimal time because we will focus only on details. We can then consider that the results in this report are a good approximation of reality.